# Uncertainty principle for the anti-commutator

J. Usera*

*Universidad Complutense de Madrid*

Quantum Mechanics makes wide use of non-commuting dynamical variables. A famous consequence of this is Heisenberg's inequality which, in its most general form, is a constraint on dispersions of any two quantum variables, that finds its best-known form for $X$ and $P_x$. We recall the general form for any two variables, given in terms of the commutator. The simple observation that any two dynamical variables generate both a commutator and an anti-commutator immediately suggests how to generalize the proof for the latter. The resulting constraint turns out to be irrelevant for the macroscopic limit and for most typical quantum cases, but not all; and, very significantly, not for the singlet state.

## I. INTRODUCTION

The general form of Heisenberg's inequalities is,

$$\triangle_{\psi} A \, \triangle_{\psi} B \geq \frac{1}{2} \left| \langle \psi \left| [A, B] \right| \psi \rangle \right| \tag{1}$$

The proof is well known —see, e.g., (Galindo-Pascual, 1989)—, but we are going to summarize it here very briefly. We will carry it out with a little more detail for the anti-commutator.

We construct a quadratic polynomial in $\lambda \in \mathbb{R}$. Namely,

$$y(\lambda) \stackrel{\text{def}}{=} \left\| (A' + i\lambda B') \left| \psi \right\rangle \right\|^2 \geq 0$$

where,

$$A' \stackrel{\text{def}}{=} A - \langle A \rangle_{\psi} \tag{2}$$

$$B' \stackrel{\text{def}}{=} B - \langle B \rangle_{\psi} \tag{3}$$

This immediately leads to (1) if $y(\lambda) \geq 0$ is to be satisfied for all $\lambda$ real. We must remember $i[A, B]$ is always Hermitian, so that $\langle \psi \left| i[A, B] \right| \psi \rangle$ is always real. Heisenberg's inequality is the condition that $y(\lambda)$ does not have non-zero real roots (the discriminant of the quadratic inequality is $\leq 0$) so that varying $\lambda$ does not allow for $y(\lambda)$ to cross to the $y < 0$ $(\lambda, y)$ half-plane.

## II. THE UNCERTAINTY PRINCIPLE FOR THE ANTI-COMMUTATOR

It should be clear before we start, that the content of the uncertainty principle depends in general *both* on the choice of $A$ and $B$, and on the choice of $|\psi\rangle$. Let us define,

$$x(\lambda) \stackrel{\text{def}}{=} \left\| (A' + \lambda B') \left| \psi \right\rangle \right\|^2 \geq 0$$

It is not difficult to prove from (2) and (3) that,

$$x(\lambda) = (\triangle_{\psi} A)^2 + \lambda^2 (\triangle_{\psi} B)^2 + \lambda \left( \langle AB \rangle_{\psi} + \langle BA \rangle_{\psi} - 2 \langle A \rangle_{\psi} \langle B \rangle_{\psi} \right) \geq 0$$

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*Electronic address: `jose.usera@telefonica.net`

For this to be satisfied for all $\lambda$ real, the inequality,

$$\left(\langle AB\rangle_{\psi}+\langle BA\rangle_{\psi}-2\langle A\rangle_{\psi}\langle B\rangle_{\psi}\right)^{2}-4\left(\triangle_{\psi}A\right)^{2}\left(\triangle_{\psi}B\right)^{2}\leq 0$$

must hold. This has a form very similar to Heisenberg's, but quick checks in most usual cases show that it does not produce upper bounds to it—in other words, it is considerably *less* restrictive. That is probably why it has never been needed or noticed. Notice that a smaller lower bound than Heisenberg's would be ineffectual; while a bigger lower bound would imply that uncertainty is actually a little bit worse than Heisenberg famously proclaimed. We will plunge directly into a particularly interesting case in which it seems to have consequences. Let us write it down in its final form,

$$\triangle_{\psi}A\,\triangle_{\psi}B\geq\frac{1}{2}\left|\langle AB\rangle_{\psi}+\langle BA\rangle_{\psi}-2\langle A\rangle_{\psi}\langle B\rangle_{\psi}\right| \tag{4}$$

Or, if you prefer,

$$\triangle_{\psi}A\,\triangle_{\psi}B\geq\frac{1}{2}\left|\langle\psi\left|\{A,B\}\right|\psi\rangle-2\langle A\rangle_{\psi}\langle B\rangle_{\psi}\right| \tag{5}$$

You can notice immediately how, when the algebra is commuting, for example in the classical domain in which all commutator or anticommutator terms of order $\hbar$ are negligible, the term inside the absolute value in expression (4) reduces to,

$$\left|\langle AB\rangle_{\psi}-\langle A\rangle_{\psi}\langle B\rangle_{\psi}\right|$$

So, as it was the case with good old Heisenberg's uncertainty principle, this one poses no significant constraints in the macroscopic limit. The expression is completely classical, and vanishes in particular whenever $A$ and $B$ are independent: it reduces to the familiar correlations of classical probability theory. Neither does it affect canonical couplings like the paradigmatic $x$, $p_x$ relation, as, if you allow some preliminary conceptually loose notions which really do not matter too much in preparation for our more precise argument, $x$ and $p_x$ are in a sense both *strongly non-commuting* ($\propto I$) and *strongly non-anti-commuting* ($\propto 2D+I$, where $D$ is the dilatation operator and $I$, the identity operator). It is in the *micro*scopic limit (and especially when $A$ and $B$ are anti-commuting: just plain anti-commuting *is* very strongly anti-commuting) where it wreaks havoc, as we are going to see.

## III. SPIN 1/2-SINGLET

This is self-explanatory: $A=\boldsymbol{n}\cdot\boldsymbol{\sigma}$, $B=\boldsymbol{m}\cdot\boldsymbol{\sigma}$

$$\{A,B\}=n_{i}m_{j}\{\sigma_{i},\sigma_{j}\}=n_{i}m_{j}2\delta_{ij}=2\boldsymbol{n}\cdot\boldsymbol{m}$$

$$\triangle_{|\psi\rangle}\left(\boldsymbol{n}\cdot\boldsymbol{\sigma}\right)\,\triangle_{|\psi\rangle}\left(\boldsymbol{m}\cdot\boldsymbol{\sigma}\right)\geq\left|\boldsymbol{n}\cdot\boldsymbol{m}-\langle\boldsymbol{n}\cdot\boldsymbol{\sigma}\rangle_{|\psi\rangle}\langle\boldsymbol{m}\cdot\boldsymbol{\sigma}\rangle_{|\psi\rangle}\right|$$

For the singlet state ($\boldsymbol{n}$, $\boldsymbol{m}$ here refer to one particle of the singlet):

$$\triangle_{|\psi\rangle}\left(\boldsymbol{n}\cdot\boldsymbol{\sigma}\right)\,\triangle_{|\psi\rangle}\left(\boldsymbol{m}\cdot\boldsymbol{\sigma}\right)\geq\left|\boldsymbol{n}\cdot\boldsymbol{m}\right|=\left|\cos\left(\boldsymbol{n},\boldsymbol{m}\right)\right|$$

What does the uncertainty principle for the commutator say?

$$[A,B]=n_{i}m_{j}[\sigma_{i},\sigma_{j}]=n_{i}m_{j}2i\epsilon_{ijk}\sigma_{k}=2i\boldsymbol{n}\wedge\boldsymbol{m}$$

$$\triangle_{|\psi\rangle}\left(\boldsymbol{n}\cdot\boldsymbol{\sigma}\right)\,\triangle_{|\psi\rangle}\left(\boldsymbol{m}\cdot\boldsymbol{\sigma}\right)\geq\left|\boldsymbol{n}\wedge\boldsymbol{m}\right|=\left|\sin\left(\boldsymbol{n},\boldsymbol{m}\right)\right|$$

An immediate corollary of this is:

$$2\left[\triangle_{|\psi\rangle}\left(\boldsymbol{n}\cdot\boldsymbol{\sigma}\right)\ \triangle_{|\psi\rangle}\left(\boldsymbol{m}\cdot\boldsymbol{\sigma}\right)\right]^2\geq\cos^2\left(\boldsymbol{n},\boldsymbol{m}\right)+\sin^2\left(\boldsymbol{n},\boldsymbol{m}\right)=1$$

with which:

$$\triangle_{|\psi\rangle}\left(\boldsymbol{n}\cdot\boldsymbol{\sigma}\right)\ \triangle_{|\psi\rangle}\left(\boldsymbol{m}\cdot\boldsymbol{\sigma}\right)\geq\frac{1}{\sqrt{2}}$$

As the axes of both vectors are, after all, arbitrary and the bound does no longer depend on them, we deduce in particular (by choosing $\boldsymbol{n}=\boldsymbol{m}$) that:

$$\triangle_{|\psi\rangle}\left(\boldsymbol{n}\cdot\boldsymbol{\sigma}\right)\geq 2^{-1/4}\simeq 0.84 \tag{6}$$

for any direction $\boldsymbol{n}$. This is an odd result. It seems to be telling us that spin for a $\frac{1}{2}\oplus\frac{1}{2}$ system in the singlet state *cannot be ascertained at all:* The error is the same order of magnitude than what one is measuring in the first place (it is actually about 0.34 bigger than 0.5=1/2). This really is not *that* surprising: The very thing we are ascribing to spin in the experiment could actually be made up of orbital angular momentum of the system; or vice-versa: Orbital angular momentum could have been called into play due to the respective linear momenta of the components of the singlet not lying on the same straight line, thus contributing an overall non-zero orbital angular momentum. In other words: It is not spin what one is measuring in $2^{\text{nd}}$-kind measurements (we are thinking of Stern-Gerlach devices) from, e.g., a decaying singlet state, but the confirmation of a linear momentum variable compatible with an initial configuration of spin that had zero overall angular momentum (spin+orbital). What guarantees this striking situation is that the total angular momentum is exactly zero and with zero dispersion for any pair of measurements of the same projection of angular momentum: correlations are there irrespective of what type of angular momentum we are measuring. This is a consequence of the well-known and often forgoten connection between isotropy of space with respect to all points and homogeneity.

Many otherwise rather careful analyses seem to forget this elementary consequence of quantum mechanics: The total state is a product of a spatial part and another part tagged with internal quantum numbers. In fact, in a more careful description than is customary, one should account for the singlet decay by writing down the corresponding superposition:

$$|\psi_{\boldsymbol{S}=0;\boldsymbol{J}=0}\rangle+\sum_{L}|\psi_{\boldsymbol{S}=\boldsymbol{L};\boldsymbol{J}=-\boldsymbol{L}}\rangle$$

But this state has a spatial part for which any internal quantum numbers go in for the ride. This whole question, we believe, has escaped many analyses of spin correlations because people insist on writing the state in a spin-only "stripped down" form: $|+-\rangle-|-+\rangle$, which is *manifestly incomplete* from the physical point of view.

## IV. CONCLUSIONS

What are the consequences of this? Is it perchance impossible to determine a spin projection in an arbitrary spin state? Of course not, as it is very easy to check for several cases, all giving zero for the bound (lower than the more restrictive given by (1)). This is a very special constraint of the singlet state in combination with the restrictions imposed by the anti-commutator. An eigenstate of an isolated particle does not produce such constraint. The experiment immediatly suggests itself of performing consecutive measurements of angular momentum; the first one filtering (spin-determining), and the second one space-time (Stern-Gerlach); and thus allegedly "contaminated" with orbital angular momentum if our reasoning is correct, to check for correcteness of (6) that fine experimental analysts should be able to devise in more precise terms. An even more tantalazing option could be to put the single state under further "pressure", by combining a filtering test for particle 1 with a space-time (Stern-Gerlach) test for particle 2; an operation that cannot be switched for 2-1, as particle 2 would no longer be travelling in the reference direction previous to the check.

### References

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